Particle Distribution Functions

MICE measured the position and momentum particle by particle, with the beam being assembled from these individual measurements. On the right shows the Monte Carlo measurements at the upstream and downstream reference planes for when no absorber was present and when a polyethylene wedge was placed between the reference planes.

Both samples had the same input beam with a nominal transverse emittance of 6 mm and 140 MeV/c momentum. However a cut on transmission has been made (insisting on the particle being transmitted to TOF2) to allow for the comparison of the upstream and downstream phase space densities. This results in slightly different input distributions for each case as the wedge may scatter out some particles beyond the aperture of the experiment.

The bias created by both the transmission losses as well as the resulting change in the particle distribution functions have not been accounted on the phase space density calculations.

Estimating the Phase Space Density

A particle beam can be described by the distribution of the particles in the beam also known as the phase space density $\rho(x,y,z,p_x,p_y,p_z)$. Liouville’s theorem states that the density of particles in phase space is a constant i.e. $\frac{\partial \rho}{\partial t} = 0$ (providing there are no dissipative forces), where the number of particles in that phase-space volume is given by:

$$N = \int \rho(x,y,z,p_x,p_y,p_z) dV$$

The $x,y,p_x,p_y,p_z$ components used to calculate the density are the measurements at the reference planes, the $z$ component calculated from the difference in arrival time at the reference plane and the mean arrival time.

Kernel Density Estimation (KDE) has been used to determine the probability of a particle having a particular phase space density in a given distribution. KDE is a non-parametric density estimation technique which makes fewer assumptions about the underlying distribution. This is done by calculating the kernel, a multivariate Gaussian centred on each data point. All of the kernels are then summed to arrive at the KDE. The KDE is then given by:

$$\rho(x) = \frac{1}{n} \sum_{i=1}^{n} K\left(\frac{1}{h} \cdot (x - X_i)\right)$$

where $K$ is the kernel function used as a function of reference point $x$, $X_i$ is the $i^{th}$ data point $X_i$ and kernel width $h$. The dimensionality is given by $d$, with sample size $n$.

Change in Phase Space Density

On the left are shown the change in the Monte Carlo Truth phase space densities (assuming the 6D density can be separated into transverse and longitudinal components) for the No Absorber and Wedge cases between the reference planes.

For the No Absorber case, the density remains conserved (6D in particular), while the Wedge shows a decrease in longitudinal density and a slight increase in transverse density. Note, the calculated densities may not including all the heated particles outside of the aperture of the experiment.

The measurements are taken at the reference planes, longitudinal z-direction (space-like) planes and not at a point in time (time-like plane). When those two planes no longer coincide due to the dispersion created at the wedge, an apparent decrease in the 6D density can be seen, as seen by the evolution of the 6D density through the 10 tracking stations (right) from Upstream Tracker Station 5 through to Downstream Tracker Station 5.

Conclusion

The conservation of the 6D phase space density in the No Absorber case shows the capability of this analysis. The Emittance Exchange effect, as to be used for a Muon Collider or Neutrino Factory, can then be quantified once corrections are made to achieve a time-like state as well as the corrections for the transmission losses.